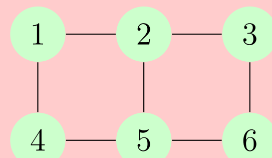
MATH 579: Combinatorics

Exam 7 Solutions

1. Find the automorphism group of the following graph. Give all elements explicitly, in standard (cycle) form.



Vertex 1 must go to one of $\{1, 3, 4, 6\}$; this choice determines all the others. Hence the automorphism group is: $\{id, (1, 3)(4, 6), (1, 4)(2, 5)(3, 6), (1, 6)(2, 5)(3, 4)\}$.

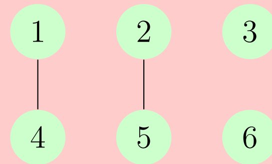
2. In S_4, set $\pi = (1, 2, 3), \tau = (2, 3, 4)$. Find all elements in the subgroup $\langle \pi, \tau \rangle$. Give all elements explicitly, in standard (cycle) form. HINT: There are fewer than 24 elements.

We get $id, \pi = (1, 2, 3), \pi^2 = (1, 3, 2), \tau = (2, 3, 4), \tau^2 = (2, 4, 3)$. We also have $\pi \circ \tau = (1, 2)(3, 4)$ and $\tau \circ \pi = (1, 3)(2, 4)$. We have $\pi^2 \circ \tau = (1, 3, 4)$ and $\tau \circ \pi^2 = (1, 4, 2)$. We have $\tau^2 \circ \pi = (1, 4, 3)$ and $\pi \circ \tau^2 = (1, 2, 4)$. Lastly, we have $\pi \circ \tau^2 \circ \pi = (1, 4)(2, 3)$. This is twelve, and if there were any more there must be 24 by Lagrange's theorem (and the hint says there isn't 24). Hence this is all of them.

3. Use Burnside's Lemma to compute the number of ways to color the vertices of K_5, distinct up to automorphism, drawn from 3 possible colors.

We list the permutations in S_5 by cycle structure, and count:
Five one-cycles: just id, 1 such.
One five-cycle: $4! = 24$ ways to order the elements (smallest one is first, order the others).
One two-cycle, three one-cycles: $\binom{5}{2} = 10$ ways to pick the elements for the two-cycle.
One three-cycle, two one-cycles: $\binom{5}{3} = 10$ ways to pick the elements for the three-cycle, $2! = 2$ ways to order them, total of 20.
One three-cycle, one two-cycle: Still 20, as in previous.
One four-cycle, one one-cycle: $\binom{5}{4} = 5$ ways to pick the elements for the four-cycle, $3! = 6$ ways to order them, total of 30.
Two two-cycles, one one-cycle: $\binom{5}{2} = 10$ ways to pick the first two-cycle, $\binom{3}{2} = 3$ ways to pick the second two-cycle. However these 30 ways overcounts, since $(1, 2)(3, 4) = (3, 4)(1, 2)$, so there are only 15.
Double-checking, $1 + 24 + 10 + 20 + 20 + 30 + 15 = 120 = 5!$, as expected.
Burnside's lemma gives $\frac{1}{120}(3^5 + 24 \cdot 3^1 + 10 \cdot 3^4 + 20 \cdot 3^3 + 20 \cdot 3^2 + 30 \cdot 3^2 + 15 \cdot 3^3) = \frac{2520}{120} = 21$.

4. Use Burnside's Lemma to compute the number of ways to color the vertices of the following graph, distinct up to automorphism, drawn from 4 possible colors.



We begin by writing the automorphism group: $id, (1,4), (2,5), (1,4)(2,5), (1,2)(4,5)$, $(1,5)(2,4)$, $(1,2,4,5), (1,5,2,4)$ and with $(3,6)$ included: $(3,6), (1,4)(3,6), (2,5)(3,6)$, $(1,4)(2,5)(3,6), (1,2)(3,6)(4,5), (1,5)(2,4)(3,6), (1,2,4,5)(3,6), (1,5,2,4)(3,6)$. There are 16 altogether.

Hence, Burnside's lemma gives $\frac{1}{16}(4^6+4^5+4^5+4^4+4^4+4^4+4^3+4^3+4^5+4^4+4^4+4^3+4^3+4^3+4^2+4^2) = \frac{8800}{16} = 550$.

5. Use Burnside's Lemma to compute the number of ways to color the vertices of C_{12}, distinct up to automorphism, drawn from 2 possible colors.

There are 12 rotations. Ten of them are given by $(1,2,3,4,5,6,7,8,9,10,11,12)$, $(1,3,5,7,9,11)(2,4,6,8,10,12)$, $(1,4,7,10)(2,5,8,11)(3,6,9,12)$, $(1,5,9)(2,6,10)(3,7,11)(4,8,12)$, $(1,6,11,4,9,2,7,12,5,10,3,8)$ and their inverses. There is also id and $(1,7)(2,8)(3,9)(4,10)(5,11)(6,12)$, which are each their own inverses.

There are also 12 reflections. Six of them pass through two opposite vertices, giving two cycles of length 1 and five of length 2. The other six reflections pass through two opposite edges, giving six cycles of length 2.

Burnside's lemma gives $\frac{1}{24}(2\cdot 2^1+2\cdot 2^2+2\cdot 2^3+2\cdot 2^4+2\cdot 2^1+2^{12}+2^6+6\cdot 2^7+6\cdot 2^6) = \frac{5376}{12} = 448$.

6. Let p be an odd prime. Use Burnside's Lemma to compute the number of ways to color the vertices of C_p, distinct up to automorphism, drawn from n possible colors.

There are p rotations: $id, \pi = (1,2,3,\ldots,p), \pi^2, \ldots, \pi^{p-1}$. Each rotation (other than id) is a single cycle of length p, because it is not possible to break up p into equal parts any other way. There are also p reflections: each line of reflection passes through one vertex and one edge (since p is odd). Hence each reflection contains one cycle of length one and $\frac{p-1}{2}$ cycles of length 2. Hence, Burnside's lemma give $\frac{1}{2p}(n^p + (p-1)n^1 + pn^{(p+1)/2})$.

NOTE: One consequence of this result is that the final expression is an integer. Hence, $n^p+(p-1)n^1+pn^{(p+1)/2}$ is a multiple of p. Some of those terms are obviously multiples of p, and may be subtracted. We conclude that $n^p - n$ is a multiple of p. Hence, this proves Fermat's Little Theorem.